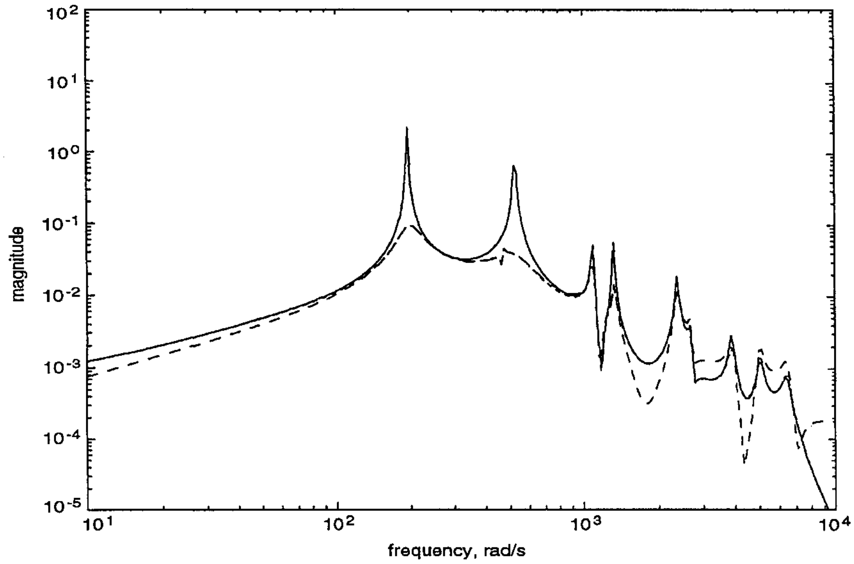


Fig. 3 Magnitudes of open-loop transfer function (——) and of the H_∞ closed-loop systems: – – –, with the input shaping filter, and - - - -, with the scaled input matrix.

the filter is represented by the dotted line. The equivalent structure with the filter was obtained by scaling the disturbance input, according to Eq. (6), and the magnitude of its transfer function is shown in Fig. 2, as a dashed line. It is clear from that figure that the structure with the filter and the structure with the scaled disturbance input have very similar frequency characteristics, and their norms are $\|G\|_\infty = 2.6903$ and $\|G\|_2 = 453.2945$ for the structure with the filter and $\|G\|_\infty = 2.6911$ and $\|G\|_2 = 453.5661$ for the structure with the scaled disturbance input.

Two frequency weighted H_∞ controllers for this structure were designed. The first one is based on the structure with a filter, whereas the second is based on the structure with scaled input matrix. The open- and closed-loop transfer functions are shown in Fig. 3. The closed-loop performance of the structure with the filter and that with the scaled input is almost identical. The closed-loop H_∞ norms are as follows: $\|G_{\mathrm{cl}}\|_\infty = 0.09681$ for the structure with the filter and $\|G_{\mathrm{cl}}\|_\infty = 0.09676$ for the structure with the scaled disturbance input. In a similar manner, H_2 controllers were designed. The closed-loop H_2 norms are as follows: $\|G_{\mathrm{cl}}\|_2 = 108.6295$ for the structure with the filter, and $\|G_{\mathrm{cl}}\|_2 = 108.7181$ for the structure with the scaled disturbance input.

Conclusions

It has been shown that for flexible structures the frequency shaping of the system properties with input (output) filters is equivalent to the scaling the modal input (output) matrix of the plant. This approach simplifies the controller design process. Instead of introducing new state variables, one modifies the gains of the modal input matrix. This is possible because the modal states related to the gains are weakly coupled, such that the modification of one state (or one gain) weakly influence the others. In addition, physical interpretation of the states remains unchanged and is related to the corresponding gains.

Acknowledgment

The research described in this paper was performed by the Jet Propulsion Laboratory, California Institute of Technology, under a contract with NASA.

References

[1]Voth, C. T., Richards, K. E., Jr., Schmitz, E., Gehling, R. N., and Morgenthaler, D. R., "Integrated Active and Passive Control Design Methodology for the LaRC CSI Evolutionary Model," NASA CR 4580, 1994.

[2]Gawronski, W., *Balanced Control of Flexible Structures*, Springer, London, 1996.

[3]Lim, K. B., and Balas, G. J., "Line-of-Sight Control of the CSI Evolutionary Model: μ Control," *Proceedings of the American Control Conference*, Chicago, IL, 1992.

Sliding Mode Controllers for Uncertain Systems with Input Nonlinearities

Kou-Cheng Hsu*
Fu-Jen University,
Taipei 24205, Taiwan, Republic of China

I. Introduction

THE stabilization problem of control systems with nonlinearities in the input has become a subject of keen interest in recent years, because nonlinearities inherently arise from practical actuators in system realization, for example, saturation, quantization, backlash, deadzone, and so on. The existence of nonlinear inputs is a source of degradation or, worse, shows instability in the performance of the system. Consequently, analysis of the problem of stability in control system design that accounts for nonlinearities in the input has become a subject of concern in research.[1–5] In addition to nonlinearities in the input, there are plant uncertainties, which originate from various sources, such as variation of plant parameters, inaccuracy arising from identification, etc. Therefore, the analysis of the problem of stability of robust control systems with plant uncertainties is as important as the problem of nonlinearities in the input.

In a robust control system, sliding mode control (SMC) is frequently adopted due to its inherent advantages of fast response and insensitivity to plant parameter variation and/or external perturbation. However, thus far, all of the published SMC studies concentrate on systems linear in the control input. The study of SMC for systems nonlinear in the control input has not been reported in the literature. In this study, a new SMC law is designed to ensure the global reaching condition of the sliding mode for uncertain systems with series nonlinearities. An example is given to verify the validity of the developed sliding mode controller.

Received July 2, 1997; revision received Feb. 27, 1998; accepted for publication March 2, 1998.

*Associate Professor, Department of Electronic Engineering, Hsinchuang.

II. Problem Formulation

The general description of an uncertain system with nonlinear input is given by

$$\dot{x}(t) = Ax(t) + B\Phi[u(t)] + f(x, p, t) \tag{1}$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^m$, $p(t) \in \mathcal{R}^q$, and $f(t) \in \mathcal{R}^s$ are the state variable, control input, uncertain parameter, and external perturbation, respectively. $A \in \mathcal{R}^{n\times n}$ is the state matrix, $B \in \mathcal{R}^{n\times m}$ is the input matrix, and $\Phi(u) \in \Psi$ is the nonlinear input function, which will be defined later. It is also assumed that, for any initial condition $x(t_0) = x_0 \in \mathcal{R}^n$, parameter $p(t) \in \mathcal{R}^q$, and control input $u(t) \in \mathcal{R}^m$, there exists a unique $x(t; x_0, p, u)$ that satisfies the system described in Eq. (1).

To delineate the nonlinear input function Φ, the following definitions are required.

Definition 1: Let diagonal matrix $\Gamma = \text{diag}[\gamma_1, \ldots, \gamma_m] \in R^{m\times m}$ be positive definite.

Definition 2: The allowed series nonlinearities $\Phi(\cdot)$ belong to the set

$$\Psi \triangleq \left\{\Phi : \mathcal{R}^m \to \mathcal{R}^m;\ \gamma_i u_i^2 \le u_i \Phi_i(u), u_i \in \mathcal{R}, i = 1, \ldots, m, u \in \mathcal{R}^m\right\} \tag{2}$$

Remark 1: For the nonlinearity $\Phi(u) \in \Psi$, $\Phi(u)$ satisfies

$$u^T\Phi(u) \ge u^T\Gamma u \ge \gamma u^T u \tag{3}$$

where $\gamma = \min\{\gamma_1, \ldots, \gamma_m\}$ and u^T is the transpose of u. For system (1), the following assumptions are made throughout.

Assumption 1: For all $i = 1, \ldots, m$, if $u_i = 0$, then $\Phi_i(u) = 0$.

Assumption 2: For the uncertain system (1), matrix pair (A, B) is controllable.

Assumption 3: For the external disturbance of the system, $f(x, p, t)$ meets the following matching condition:

$$f(x, p, t) = Be(x, p, t), \qquad \|e(x, p, t)\| \le \alpha\|x\| + \beta$$
$$\forall (x, p, t) \in \mathcal{R}^n \times \mathcal{R}^q \times \mathcal{R} \tag{4}$$

where α and β are known nonnegative constants and $\|W\|$ is the Euclidean norm when W is a vector or the induced norm when W is a matrix.

III. Controller Design

A systematic procedure for sliding mode controller design of uncertain nonlinear control systems is presented. There are two major phases in SMC design: 1) to select an appropriate switching surface for the system to ensure that the sliding motion on the switching hyperplane possesses the desired properties and 2) to determine a law for the switching control that is able to force the system moving toward the sliding surface and staying on it. First, the switching surface is defined as

$$\sigma(t) = \mathcal{K}x(t) = 0 \tag{5}$$

where $\mathcal{K} \in \mathcal{R}^{m\times n}$ is a constant matrix that needs a nonzero determinant of the product matrix $(\mathcal{K}B)$. As long as the sliding mode $\sigma(t) = 0$ results, it is always accompanied by the condition $\dot{\sigma}(t) = 0$. Therefore, the following equivalent control Φ_{eq} in the sliding mode can be derived from $\mathcal{K}\dot{x}(t) = 0$:

$$\Phi_{\text{eq}} = -(\mathcal{K}B)^{-1}\mathcal{K}(Ax + Be) \tag{6}$$

Remark 2: Note that the equivalent control Φ_{eq} is a mathematical tool derived for the analysis of sliding motion rather than a real control law that can be generated in practical systems. Also note that the equivalent control generates an ideal sliding motion on the switching surface, whereas the real SMC generates a trajectory close to the ideal sliding motion around the switching surface.

Consequentially, the equivalent dynamic system with nonlinear input in the sliding mode is given as

$$\dot{x}(t) = [I - B(\mathcal{K}B)^{-1}\mathcal{K}]Ax(t) \tag{7}$$

where I is an $n \times n$ identity matrix. However, for the nominal system $\dot{x}(t) = Ax(t) + B\Phi[u(t)]$, its equivalent control $\Phi_{\text{eq}} = -(\mathcal{K}B)^{-1}\mathcal{K}Ax$ yields the same equivalent dynamic system (7) in the sliding mode. It can be seen that the system dynamics is dominated by the sliding motion, and the invariance condition holds for the system with nonlinear input. From the preceding analysis, it can be seen that uncertain systems possess the same properties in the sliding mode irrespective of whether or not there are input nonlinearities. Thereby, selection of a switching surface can go through the same process of SMC design for systems with linear input, for example, as given in Refs. 6 and 7.

Once a proper switching plane has been chosen, it is followed by choosing a discontinuous control law to complete the sliding mode controller design. Before constructing the desired controller, the following lemma has to be established.

Lemma 1: For all allowable nonlinearities Φ_i belonging to set Ψ in Eq. (2), there exists a known continuous function $\rho(\cdot) : R_+ \to R_+$, $\rho(0) = 0$, $\rho(p) \ge 0$ for $p \ge 0$, such that, for all $u \in R^m$,

$$\rho(\|u\|) \le u^T\Phi(u) \tag{8}$$

and there also exists a continuous function $\phi(\cdot) : R_+ \to R_+$, $\phi(0) = 0$, $\phi(p) \ge 0$ for $p \ge 0$, such that, for all $q \ge 0$,

$$\rho(\phi(q)) \ge q\phi(q) \tag{9}$$

Proof: We can choose $\rho(p) = \gamma p^2$ with $\gamma > 0$, then

$$\rho(\|u\|) = \gamma\|u\|^2 = \gamma u^T u \le u^T\Phi(u) \tag{10}$$

Next, if we take $\phi(q) = \delta q$ with $\delta \ge 1/\gamma$, then

$$\rho(\phi(q)) = \gamma\phi^2(q) = \gamma\delta q\phi(q) \ge q\phi(q) \tag{11}$$

□

To obtain asymptotical stability of the sliding mode, the desired controller is suggested by

$$u(t) = -\frac{B^T\mathcal{K}^T\sigma(t)}{\|B^T\mathcal{K}^T\sigma(t)\|}\phi(q) \tag{12}$$

where $\phi(q) = \delta q$ with $\delta \ge 1/\gamma$, and function $q(\cdot) : R^n \times R \to R$ is defined as

$$q(x, t) = \eta\{[\|(\mathcal{K}B)^{-1}\mathcal{K}A\| + \alpha]\|x\| + \beta\}, \qquad \eta > 1 \tag{13}$$

Note that $\|u(t)\| = \phi(q)$, which is implied in Eq. (12). In the following theorem, the developed control law (12) will be proven to be able to drive the uncertain system trajectory into the sliding mode $\sigma(t) = 0$.

Theorem 1: Let us consider the uncertain system (1) with series nonlinearities that is subjected to assumptions 1–3. If the applied input $u(t)$ to the system is defined by Eq. (12), then the sliding motion of the system trajectory is guaranteed to be asymptotically stable.

Proof: Let $V = \frac{1}{2}\|\sigma(t)\|^2$ be the Lyapunov function candidate of system (1), then its time derivative $\dot{V} = \sigma^T(t)\dot{\sigma}(t)$. Now consider the hitting condition of the sliding mode in Eq. (5). By substituting Eq. (1) into the time derivative $\dot{x}(t)$ in $\dot{\sigma}(t)$, we get the following result:

$$\sigma^T(t)\dot{\sigma}(t) = \sigma^T(t)\mathcal{K}\{Ax(t) + B\Phi[u(t)] + Be(x, p, t)\} \tag{14}$$

Then applying Eq. (4) and $\|AB\| \le \|A\|\|B\|$ to Eq. (14), we get

$$\sigma^T(t)\dot{\sigma}(t) \le \|\sigma^T\mathcal{K}B\|\|(\mathcal{K}B)^{-1}\mathcal{K}A\|\|x\|$$
$$+ \sigma^T\mathcal{K}B\Phi(u) + \|\sigma^T\mathcal{K}B\|(\alpha\|x\| + \beta) \tag{15}$$

By combining Eqs. (3) and (12), we get

$$u^T\Phi(u) = -\frac{\sigma^T\mathcal{K}B}{\|B^T\mathcal{K}^T\sigma\|}\phi(q)\Phi(u) \geq \gamma u^T u = \rho(\|u\|) \qquad (16)$$

Because $\|u\| = \phi(q)$, from lemma 1, we obtain

$$u^T\Phi(u) \geq \rho(\|u\|) = \rho(\phi(q)) \geq q\phi(q) \qquad (17)$$

Therefore, from Eqs. (16) and (17) the following expression can be derived:

$$\sigma^T\mathcal{K}B\Phi(u) \leq -q(x,t)\|B^T\mathcal{K}^T\sigma\| \qquad (18)$$

By placing Eq. (18) into Eq. (15), we get

$$\sigma^T(t)\dot\sigma(t) \leq \|\sigma^T\mathcal{K}B\|\{\|(\mathcal{K}B)^{-1}\mathcal{K}A\|\|x\| - q(x,t) + [\alpha\|x\| + \beta]\}$$
$$= (1-\eta)\|\sigma^T\mathcal{K}B\|\{[\|(\mathcal{K}B)^{-1}\mathcal{K}A\| + \alpha]\|x\| + \beta\} \qquad (19)$$

Because $\eta > 1$ has been assumed in Eq. (13), one can directly conclude that $\sigma^T(t)\dot\sigma(t) < 0$. Thus, the proof is achieved completely. □

Remark 3: For a case where $\Phi(u)$ denotes a sector-bounded vector, the proposed SMC (12) still works effectively for systems with sector-bounded nonlinear inputs. Let $\Phi[u(t)] = \{\Phi_1(u), \ldots, \Phi_m(u)\}$, where $\Phi_i(u)$ for $i = 1, \ldots, m$ is characterized by

$$\gamma_{Li}u_i^2 \leq u_i\Phi_i(u) \leq \gamma_{Hi}u_i^2 \qquad (20)$$

Then we have $\gamma_L u^T u \leq u^T\Phi(u) \leq \gamma_H u^T u$, where $\gamma_L = \min\{\gamma_{L1}, \ldots, \gamma_{Lm}\}$ and $\gamma_H = \max\{\gamma_{H1}, \ldots, \gamma_{Hm}\}$. If we let $\gamma_L = \gamma$ and $\gamma_H \to \infty$, then the sector function $\Phi(u)$ satisfies Eq. (3); that is, the upper bound of the sector nonlinearities can be released. Therefore, Eq. (12) is applicable to the system with sector-bounded inputs.

Remark 4: If $\Phi(u) = u$, which means that the system has no nonlinear inputs, then the proposed SMC law in Eq. (12) is still applicable to control such a system as long as $\gamma \leq 1$, because $u^T\Phi(u) = u^T u \geq \gamma u^T u$ is always satisfied.

IV. Illustrative Example

Consider the following uncertain systems with/without nonlinear inputs. The initial condition $x(0) = [1 \;\; 3 \;\; -3]^T$ is arbitrarily taken for both systems.

System 1:

$$\dot x(t) = Ax(t) + B(\Phi[u(t)] + e(x,p,t))$$
$$y(t) = [1 \quad 0 \quad 0]x(t)$$

System 2:

$$\dot x(t) = Ax(t) + B[u(t) + e(x,p,t)]$$
$$y(t) = [1 \quad 0 \quad 0]x(t)$$

where

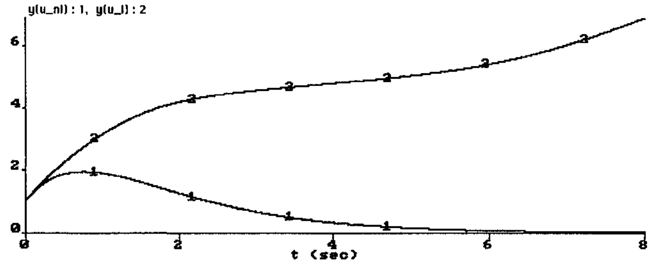
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Phi[u(t)] = \left(0.5 + 0.3\sin u(t) + e^{0.2|\cos u(t)|}\right)u(t)$$

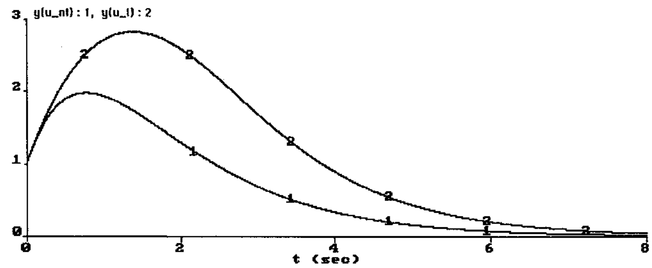
$$e(x,p,t) = (0.3 + 0.2\cos x_1)\sqrt{x_1^2 + x_2^2 + x_3^2} - 0.5\sin x_2$$

For both systems, the switching hyperplane is selected as

$$\sigma(t) = x_1(t) + 2x_2(t) + x_3(t)$$



a) System $\dot x(t) = Ax(t) + B[\Phi(u) + e]$



b) System $\dot x(t) = Ax(t) + B(u + e)$

Fig. 1 Output y controlled by $u = u_{nl}$, 1, and $u = u_l$, 2.

From Eq. (12) and by following the procedure of SMC design for a system without input nonlinearity,[7] we have

$$u_{nl} = -\frac{B^T\mathcal{K}^T\sigma(t)}{\|B^T\mathcal{K}^T\sigma(t)\|}\delta\eta\{[\|(\mathcal{K}B)^{-1}\mathcal{K}A\| + \alpha]\|x\| + \beta\}$$

$$u_l = -(\mathcal{K}B)^{-1}[\mathcal{K}Ax + \|\mathcal{K}B\|(\alpha\|x\| + \beta)\mathrm{sgn}(\sigma)]$$

where u_{nl} is designed for system 1 and u_l is designed for system 2, respectively.

On the basis of Eqs. (3) and (4), the following coefficients are taken:

$$\delta = 1.11, \qquad \alpha = 0.5, \qquad \beta = 0.5, \qquad \eta = 1.2$$

For system 1, Fig. 1a shows the transient response of output y, controlled by u_{nl} and u_l, respectively. The developed sliding mode controller u_{nl} for the nonlinear control system does work effectively; however, the sliding mode controller u_l designed for the linear control system cannot control a system with nonlinear input to obtain the desired dynamics of the system. For system 2, Fig. 1b shows the performance of output y, which is well controlled by the sliding mode controllers u_{nl} and u_l, respectively. It is obvious to see that the proposed SMC law is suitable for systems with or without series nonlinearities, whereas the traditional SMC is not applicable to a system with series nonlinearities.

V. Conclusions

A new sliding mode controller is proposed for the purpose of stabilizing uncertain systems with series nonlinearities. The study shows an SMC design to ensure the global reaching condition of the sliding mode for uncertain systems with series nonlinearities. Uncertain nonlinear control systems possess the same attractive feature of insensitivity to uncertainty and/or perturbation as do linear control systems. In addition, the developed controller can control systems with or without nonlinear inputs, which cannot be achieved by the traditional SMC designed for systems without nonlinearities.

References

[1]Ohta, Y., Gurov, I. R., and Haneda, H., "Quadratic Stabilization of Uncertain Lur'e-Postnikov Systems via Linear State Feedback," *Control Theory and Advanced Technology*, Vol. 8, No. 2, 1992, pp. 353–360.

[2]Recker, D., and Kokotovic, P. V., "Indirect Adaptive Nonlinear Control of Discrete-Time Systems Containing a Deadzone," *IEEE Proceedings of the 32nd Conference on Decision and Control* (San Antonio, TX), 1993, pp. 2647–2653.

[3]Tao, G., and Kokotovic, P. V., "Continuous-Time Adaptive Control of Systems with Unknown Backlash," *IEEE Transactions on Automatic Control*, Vol. 40, No. 6, 1995, pp. 1083–1087.
[4]Bernstein, D. S., and Haddad, W. M., "Nonlinear Controller for Positive Real Systems with Arbitrary Input Nonlinearities," *IEEE Transactions on Automatic Control*, Vol. 39, No. 7, 1994, pp. 1513–1517.
[5]Haddad, W. M., and Kapila, V., "Antiwindup Controllers for Systems with Input Nonlinearities," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 6, 1996, pp. 1387–1390.
[6]Utkin, V. I., and Young, K. D., "Methods for Constructing Discontinuous Planes in Multidimensional Variable Structure Systems," *Automatic Remote Control*, Vol. 9, No. 10, 1978, pp. 1466–1470.
[7]Shyu, K. K., and Liu, C. Y., "Variable Structure Controller Design for Robust Tracking and Model Following," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 6, 1996, pp. 1395–1397.

Quaternion Parameterization and a Simple Algorithm for Global Attitude Estimation

R. G. Reynolds*
TRW Space and Electronics Group, Chantilly, Virginia 20151

Introduction

OF the many representations that exist for attitude, quaternions are popular for several reasons.

1) The propagation differential equations have no singularities, in contrast to, e.g., Euler's angles or Gibbs' vectors,[1] which require special handling near these points.

2) Modulo a scaling factor, the quaternion is a minimal three-degree-of-freedom representation of attitude as opposed to, e.g., direction cosine matrices, which must be orthonormalized in practical applications.

3) There are a number of simple and direct methods available to implement controllers with quaternion feedback, which are globally stable.[2–4]

In this Note, we examine the decomposition of the quaternion attitude representation in terms of a basis formed using observation unit vectors and a number of direct applications of this result. In particular, we will consider the implementation of a global attitude estimation scheme in both recursive and batch form. The latter batch algorithm may be shown to be identical to the quaternion estimation (QUEST) algorithm.[5]

Representations and Conventions

Vector arrays should always be considered to be column arrays, with superscript T indicating transposition to a row vector. We define the quaternion $\boldsymbol{q} \in \boldsymbol{H}$, the division ring of 4-tuple real quaternions,[6,7] with group composition given by

$$\boldsymbol{q}_{13} = \boldsymbol{q}_{12} \circ \boldsymbol{q}_{23} = \left(q_{12}\boldsymbol{Q}_{23} + q_{23}\boldsymbol{Q}_{12} + \boldsymbol{Q}_{12} \times \boldsymbol{Q}_{23},\ q_{12}q_{23} - \boldsymbol{Q}_{12}^T\boldsymbol{Q}_{23}\right) \tag{1}$$

componentwise addition, and identity $\vartheta = (0, 0, 0, 1)$. In Eq. (1), $\times$ denotes the vector cross product, and we have introduced the shorthand representation of a quaternion as $\boldsymbol{q} = (\boldsymbol{Q}, q)$, where $\boldsymbol{Q} \in \boldsymbol{R}^3$ is the so-called vector part of the quaternion and $q \in \boldsymbol{R}$ is the scalar part.

The subscript on a given quaternion in Eq. (1) is a helpful mnemonic, which indicates the rotation that it represents, e.g., $\boldsymbol{q}_{12}$ is associated with the rotation transformation from a given reference frame 1 to reference frame 2. Note that composition takes place in such a way that the common reference frame is contiguous.

There exist maps $p: V \to \boldsymbol{R}^3$ and $i: \boldsymbol{R}^3 \to V$ termed canonical projection and inclusion, respectively, which establish a correspondence between vectors in $\boldsymbol{R}^3$ and the subset $V \subset \boldsymbol{H}$ consisting of quaternions with scalar part equal to zero. In general, the action of these operators shall be implicit. An example of this duality is the rotation of vectors in $\boldsymbol{R}^3$, which is identified with the conjugacy class of V in $\boldsymbol{H}$

$$V^* = \left\{\boldsymbol{v}_1 = \boldsymbol{q}_{12} \circ \boldsymbol{v}_2 \circ \boldsymbol{q}_{12}^{-1} \,\middle|\, \boldsymbol{q}_{12} \in \boldsymbol{H}, \boldsymbol{v}_2 \in V\right\} \tag{2}$$

Note how the subscripts line up in a consistent way in this representation.

For numerical considerations in practical applications, attention is often focused on the subgroup $\boldsymbol{U} \lhd \boldsymbol{H}$ of unit quaternions, those for which the Euclidean norm in $\boldsymbol{R}^4$ is unity, and for which inversion is trivial negation of the vector part of the quaternion. The unity magnitude of $\boldsymbol{q} \in \boldsymbol{U}$ implies $\boldsymbol{U} \sim S^3$, the three sphere endowed with the quaternion group structure. For clarity of presentation, quaternions in this Note are generally not constrained to be unit quaternions.

Quaternion Parameterization Using Observation Unit Vectors

Suppose that a unit vector $\boldsymbol{x}_1$ in a given Euclidean reference frame 1 is to be transformed to a new unit vector $\boldsymbol{x}_2$ in frame 2 via conjugation by the quaternion $\boldsymbol{q}_{12}$ as in Eq. (2). The quaternion in question might take the form of

$$\boldsymbol{q}_{12} \sim \left(\boldsymbol{x}_2 \times \boldsymbol{x}_1,\ 1 + \boldsymbol{x}_2^T\boldsymbol{x}_1\right) \tag{3}$$

where the $\sim$ symbol has been used to indicate that $\boldsymbol{q}_{12}$ is equivalent to any quaternion that may be obtained by scaling expression (3). This quaternion represents the minimum angle rotation required to transform $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$.

If we visualize the transformation process, it is the rotation about an axis orthogonal to both $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, as pictured in the top-center position in Fig. 1. Figure 1 is somewhat misleading, in that the vector part of the quaternion appears to be the negative axis of rotation. However, the quaternion represents the rotation of a frame of reference in which vector $\boldsymbol{x}_1$ appears as vector $\boldsymbol{x}_2$ following the rotation, and so the vector part of the quaternion is actually the positive axis of rotation in this sense.

Another quaternion that would rotate $\boldsymbol{x}_1$ into $\boldsymbol{x}_2$ is given by

$$\boldsymbol{q}_{12} \sim (\boldsymbol{x}_2 + \boldsymbol{x}_1, 0) \tag{4}$$

which represents a 180-deg rotation as shown in the lower-left position in Fig. 1.

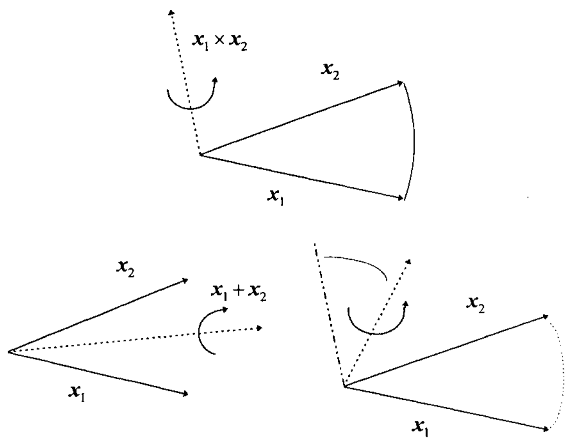


Fig. 1 Transformations.

Received June 3, 1997; revision received Feb. 27, 1998; accepted for publication March 8, 1998.

*Staff Engineer, Spacecraft Operations East, 14320 Sullyfield Circle, Suite 2100.